Chapter 6

Systems of Linear Equations

Solutions of systems of linear equations in two and three variables – Elementary row operations, Elimination method, Substitution method, Gauss-Jordan method and Matrix inversion method..

6.1 Introduction

A set of m equations in n unknowns $x_1, x_2, ..., x_n$ of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots\ \cdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the coefficients $a_{11}, a_{12}, ..., a_{mn}, b_1, ..., b_m$ are real numbers is called a system of linear equations. A solution of the system of linear equations is a vector $(x_{10}, x_{20}, ..., x_{n0})^T$ which when substituted for the unknowns satisfies all the m equations simultaneously. In general a system of linear equations may have a unique solution, or more than one solution or no solution at all.

Definition:

For the system of equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\ \vdots\quad\ \vdots\quad\ \vdots\quad\ \vdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned},$$

the $m \times n$ matrix $A = \begin{bmatrix} a_{11} & \cdots & \cdots & a_{1n} \\ & \vdots & \vdots & \\ & \vdots & \vdots & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$ is said to be the <u>*coefficient matrix*</u> of the system.

The $(n \times 1)$ vector of variable and the $(m \times 1)$ vectors of constants are respectively $x = (x_1, x_2, ..., x_n)^T$ and $b = (b_1, b_2, ..., b_m)^T$. Note that the system of equations may be represented as

$$Ax = b.$$

The matrix $\left[\underline{A} \mid \underline{b}\right] = \begin{bmatrix} a_{11} & \cdots & a_{1n} & | & b_1 \\ \vdots & \vdots & \vdots & | & \vdots \\ a_{m1} & \cdots & a_{mn} & | & b_m \end{bmatrix}$ is known as the <u>*augmented matrix*</u> of the system.

Note: As far as this course is concerned we shall only consider up to a maximum of three equations in three unknowns.

Definition:

A system of two linear equations in two variables (unknowns) may be written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned},$$

while a system of three linear equations in three variables (unknowns) is written in the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}.$$

The following theorem guarantees that there are in fact three operations that will not alter the solution(s) of a system of linear equations.

Theorem 6.1 (Without Proof)

If a system of linear equations is transformed into another linear system by one of the following operations:

(1) *Interchanging any two equations.*

(2) *Multiplying the terms of any equation by a non-zero real number.*

(3) *Adding to the terms of any equation, a scalar multiple of the corresponding terms of another equation,*

then the two systems have the same solution(s).

The following example illustrates the use of Theorem 6.1.

Example 1

Find the solution of the following system of equations.

$$\begin{aligned} 2x_1 + 2x_2 &= 4 \\ 3x_1 + 3x_2 + x_3 &= 9 \\ x_2 + x_3 &= 3 \end{aligned}$$

To illustrate that the operations (using Theorem 6.1), we first write the augmented matrix of the system.

$$\left[\begin{array}{ccc:c} 2 & 2 & 0 & 4 \\ 3 & 3 & 1 & 9 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

Now we perform the operations mentioned in Theorem 6.1, on the coefficients of the system. The augmented matrix corresponding to each step is also written. The operations permitted in the theorem can be performed correspondingly on the augmented matrix.

First obtain the coefficient 1 for x_1 in the first equation by multiplying the terms of the first equation by $1/2$.

$$\begin{aligned} x_1 + x_2 &= 2 \\ 3x_1 + 3x_2 + x_3 &= 9 \\ x_2 + x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 3 & 3 & 1 & 9 \\ 0 & 1 & 1 & 3 \end{array}\right].$$

Secondly eliminate x_1 from the other equations by adding the terms of the second equation to –3 times the first equation.

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_3 &= 3 \\ x_2 + x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 1 & 3 \end{array}\right].$$

Now that x_1 appears only in the first equation and has coefficient 1, we turn our attention to x_2. First obtain the coefficient 1 for x_2 by interchanging the second and third equations:

$$\begin{aligned} x_1 + x_2 &= 2 \\ x_2 + x_3 &= 3 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Now eliminate x_2 from the other equation by adding the terms of the first equation to -1 time the corresponding terms in the second equation.

$$\begin{array}{c} x_1 - x_3 = -1 \\ x_2 + x_3 = 3 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

Now that x_2 appears only in the second equation and has coefficient 1, we turn our attention to x_3. In the third equation, x_3 has coefficient 1. Therefore we proceed immediately to eliminate x_3 from the other equations by adding to the terms of the first equation the corresponding terms of the third equation and by adding to the corresponding terms of the second equation –1 times the corresponding terms of the third equation.

$$\begin{array}{c} x_1 = 2 \\ x_2 = 0 \\ x_3 = 3 \end{array} \qquad \left[\begin{array}{cccc} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right].$$

The solution of the latter system is (2, 0, 3) and by Theorem 6.1, the solution of the systems in each step is also (2, 0, 3).

Definition:

When two systems of linear equations have the same set of solutions, the systems are said to be equivalent.

Hence the systems in each step of the previous example are all equivalent.

6.2 Methods of Solving a System of Linear Equations

6.2.1 Elimination Method

In order to solve two independent linear equations simultaneously, we first eliminate one of the two unknowns from the two equations. We then solve the resulting equation in the other unknown. To find the eliminated unknown, we substitute the value obtained in either one of the given equations.

Example 2.

Solve the following system of linear equations by the Elimination method.

$$2x - y = 1 \qquad (1)$$
$$3x + 2y = 12 \qquad (2)$$

Solution

Let us eliminate y. We multiply equation (1) by 2 to get

$$4x - 2y = 2 \qquad (3).$$

Add equations (2) and (3) to get $7x = 14 \Rightarrow x = 2$. Substitute $x = 2$ in equation (1) to get $2(2) - y = 1 \Rightarrow y = 3$.

Hence the solution for the system of equations (1) and (2) is $x = 2$ and $y = 3$.

Example 3

Solve the following system of linear equations by the Elimination method.

$$3x - y = 1.5 \qquad (1)$$
$$6x - 2y = 3 \qquad (2)$$

Solution

Let us eliminate y. We multiply equation (1) by 2 to get $6x - 2y = 3$. Notice that this equation is the same as equation (2). Hence equations (1) and (2) are the same. From equation (1) we obtain $y = 3x - 1.5 \qquad (3)$.

For any value of x, we obtain a value of y from equation (3). We therefore conclude that there are infinitely many solutions for this system of linear equations.

Example 4

Solve the following system of linear equations using the Elimination method.

$$3x-2y=10 \qquad (1)$$

$$-6x+4y=15 \qquad (2)$$

Solution

Multiply equation (1) by 2 to get $6x-4y=20$. Add this equation to equation (2) to get $0=35$.

This is impossible. In other words there are no solutions to this sytem of equations. In such a case, we say that the system of equations is inconsistent.

Example 5

Solve the following system of linear equations by the Elimination method.

$$2x_1+4x_2+6x_3=22 \qquad (1)$$

$$3x_1+8x_2+5x_3=27 \qquad (2)$$

$$-x_1+x_2+2x_3=2 \qquad (3)$$

Solution

Step 1: Consider equations (1) and (2).

Let us eliminate x_2, multiply equation (1) by 2 to get $4x_1+8x_2+12x_3=44$. Subtract equation (2) from this equation to get

$$x_1+7x_3=17 \qquad (4)$$

Step 2: Consider equations (1) and (3).

Let us eliminate x_2. Multiply equation (3) by 4 to get $-4x_1+4x_2+8x_3=8$. Subtract equation (1) from this equation to get

$$-6x_1+2x_3=-14 \qquad (5)$$

Step 3: Consider equations (4) and (5).

They are as follows:

$$x_1+7x_3=17 \qquad (4)$$

$$-6x_1+2x_3=-14 \qquad (5)$$

Let us eliminate x_1. Multiply equation (4) by 6 to get $6x_1+42x_3=102$. Add this to equation (5) to get $x_3=\frac{88}{44}=2$. Substitute this value of x_3 into equation (4) to get $x_1+7(2)=17\Rightarrow x_1=3$. In order to get x_2, substitute $x_1=3$ and $x_3=2$ into equation (3) to get $-3+x_2+2(2)=2\Rightarrow x_2=1$. Hence the system has the solution $x_1=3$, $x_2=1$ and $x_3=2$.

6.2.2 Substitution Method

Example 6

Solve the system of linear equations using the Substitution method.

$$2x - 3y = 6 \qquad (i)$$

$$x + y = 12 \qquad (ii)$$

Solution

From (ii) we get $x=12-y$. Substituting for x in (i) we get

$$2(12-y)-3y=6\Rightarrow 24-2y-3y=6$$

$$\Rightarrow -5y=6-24\Rightarrow -5y=-18\Rightarrow y=\frac{18}{5}$$

Substituting for y in (ii) to obtain $x+\frac{18}{5}=12 \Rightarrow x=12-\frac{18}{5}=\frac{42}{5}$.

Thus the solution is $x=42/5,\ y=18/5$.

6.2.3 The Gauss-Jordan Method

The Gauss Jordan method is based on the elementary row operations. We have noticed that operations ordinarily performed on equations of a given system can be performed on the corresponding rows of the associated augmented matrix. It is a general procedure for solving the system of linear equations by performing the sequence of elementary row operations that transform the augmented matrix into its unique reduced form (or reduced echelon form).

Definition:

The following operations performed on a matrix are called *elementary row operations:*

1. *The interchange of any two rows.*
2. *The multiplication of the entries of any row by a non-zero real number.*
3. *The addition to the entries of any row, k times the corresponding entries of any other row, where k is a real number.*

Example 7. Illustrate the elementary row operations on the matrix $\underline{A}=\begin{bmatrix}2 & 3\\ 1 & 4\\ 3 & 1\end{bmatrix}$.

Solution

Consider the matrix $\underline{A}=\begin{bmatrix}2 & 3\\ 1 & 4\\ 3 & 1\end{bmatrix}$.

(i) Interchange the 2nd row and 3rd row. The notation $R_2 \leftrightarrow R_3$ will be used to denote this particular elementary row operation. The resulting matrix will be

$$\underline{B}=\begin{bmatrix}2 & 3\\ 3 & 1\\ 1 & 4\end{bmatrix}.$$

(ii) Multiply each of the entries of the 2nd row of $\underline{B}$ by 2. The notation $R_2 \rightarrow 2R_2$ will be used to denote this particular elementary row operation. The resulting matrix will be

$$\underline{C}=\begin{bmatrix}2 & 3\\ 6 & 2\\ 1 & 4\end{bmatrix}.$$

(iii) Add –2 times the entries of the first row to the corresponding entries of the 3rd row of $\underline{C}$. The notation $R_3 \rightarrow R_3-2R_1$ will be used to designate this particular elementary row operation. The resulting matrix $\underline{D}=\begin{bmatrix}2 & 3\\ 6 & 2\\ -3 & -2\end{bmatrix}$.

Example 8.

Use the Gauss-Jordan method to solve the following systems of linear equations:

1. $\begin{aligned}2x_1+3x_2&=5\\ -4x_1+5x_2&=1\end{aligned}$

2. $\begin{aligned}2x_1+4x_3&=8\\ 3x_1+x_2&=4\\ x_2+x_3&=6\end{aligned}$

Solution

1. The augmented matrix is $\left[\begin{array}{cc|c} 2 & 3 & 5 \\ -4 & 5 & 1 \end{array}\right]$.

Now, $R_1 \to (1/2)R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 3/2 & 5/2 \\ -4 & 5 & 1 \end{array}\right]$. $R_2 \to R_2 + 4R_1 \Rightarrow \left[\begin{array}{cc|c} 1 & 3/2 & 5/2 \\ 0 & 11 & 11 \end{array}\right]$

$R_2 \to (1/11)R_2 \Rightarrow \left[\begin{array}{cc|c} 1 & 3/2 & 5/2 \\ 0 & 1 & 1 \end{array}\right]$, $R_1 \to R_1 - (3/2)R_2 \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

The final matrix is the augmented matrix of the system $\left.\begin{array}{rcl} x_1 & = & 1 \\ x_2 & = & 1 \end{array}\right\}$ whose solution is (1,1) and since the original system is equivalent to the final system, it follows that (1,1) is also the solution of the original system.

2. The augmented matrix is $\left[\begin{array}{ccc|c} 2 & 0 & 4 & 8 \\ 3 & 1 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{array}\right]$

$R_1 \to (1/2)R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 3 & 1 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{array}\right]$, $R_2 \to R_2 - 3R_1 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -6 & -8 \\ 0 & 1 & 1 & 6 \end{array}\right]$

$R_3 \to R_3 - R_2 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & 7 & 14 \end{array}\right]$, $R_3 \to R_3/7 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$

$R_1 \to R_1 - 2R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -6 & -8 \\ 0 & 0 & 1 & 2 \end{array}\right]$, $R_2 \to R_2 + 6R_3 \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right]$

Therefore, the solution is given by (0, 4, 2).

6.2.4 Matrix Inversion Method

Conside the system of linear equations by expressed by

$$\mathrm{Ax} = \mathrm{b}.$$

Suppose that A^{-1} exists. Premultiplying both sides by A^{-1},we get

$$\mathrm{A}^{-1}\mathrm{Ax} = \mathrm{A}^{-1}\mathrm{b}$$
$$\mathrm{Ix} = \mathrm{A}^{-1}\mathrm{b}$$
$$\mathrm{x} = \mathrm{A}^{-1}\mathrm{b}, \text{ since } \mathrm{A}^{-1}\mathrm{A} = \mathrm{I}, \text{ Ix=x.}$$

Example 9.

Solve the system of equations

$$x_1 - x_2 = 2$$
$$x_2 + 3x_3 = 1$$
$$x_1 - 2x_2 = 7$$

We have, $A = \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 3 \\ 1 & -2 & 0 \end{array}\right] . b = \left[\begin{array}{c} 2 \\ 1 \\ 7 \end{array}\right]$

First we shall find A^{-1} by elementary row operations.
Consider the system

$$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 1 & -2 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$R_3 - R_1 \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 + R_2 \rightarrow \\ \\ R_3 + R_2 \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 3 & -1 & 1 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_3 \rightarrow \\ R_2 - R_3 \rightarrow \\ \dfrac{R_3}{3} \rightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\dfrac{1}{3} & \dfrac{1}{3} & \dfrac{1}{3} \end{array}\right]$$

Thus we get $$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & -1 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Thus $$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1}b = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 0 & -1 \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 2\times 2 + 0\times 1 + (-1)\times 7 \\ 1\times 2 + 0\times 1 + (-1)\times 7 \\ \left(\frac{-1}{3}\right)\times 2 + \left(\frac{1}{3}\right)\times 1 + \left(\frac{-1}{3}\right)\times 7 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}$$

DO IT YOURSELF EXERCISES

1. Express the following system of linear equations as a single matrix equation and define each matrix used. Also solve the system of equations using any method that you know.

$$\begin{aligned} x_1 - x_2 &= 2 \\ x_2 + 3x_3 &= 1 . \\ x_1 - 2x_2 &= 7 \end{aligned}$$

2. Solve the following systems of linear equations using all the four methods you learnt in this chapter and verify whether all the methods lead to the same set of solutions.

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(i) $\begin{aligned} 5x - 6y &= 7 \\ 2x + y &= 4 \end{aligned}$, (ii) $\begin{aligned} 3x + 2y + 4z &= 5 \\ 2x - 3y + z &= 8 \\ x + 2y - 4z &= 5 \end{aligned}$, (iii) $\begin{aligned} 4x - 6y + 10z &= 6 \\ x - 2y + 3z &= 1 \\ x - 4y &= 5 \end{aligned}$ and (iv) $\begin{aligned} y - z &= 2 \\ x + y &= 1 \\ 2x + y - z &= 6 \end{aligned}$